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On the non-linear stability of plane Couette flow

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In the present paper the stability of plane Couette flow is studied. The stream function and distribution of vorticity and the Reynolds stresses for the linearized solution are computed. The stability effect of the non-linear terms are also discussed and it is found that for small amplitudes, the non-linear terms are destabilizing. A neutral curve determining the necessary amplitude in order to get instability, is found. The convergence of the expansion in the latter case is, however, somewhat uncertain and the result should therefore only be considered as a first, rough approximation.

1. Introduction

The stability of plane Couette flow has been investigated in several papers, notably in those by Hopf (1914) (a pioneering work), Grohne (1954), Riis (1962) and Gallagher & Mercer (1962). They examine the problem by considering infinitesimal disturbances such that the equations may be linearized. The result of these papers is that an infinite set of eigensolutions exists, all of them being stable modes. There is no proof, however, that this set is complete, i.e. it is a possibility that unstable modes also exist. It seems, however, to be generally believed now that no such modes exist. It is pertinent to mention here that if the velocity of propagation of the waves (eigensolutions) is assumed to be equal to the velocity in the centre of the field, it may be shown that the Couette flow is stable (Southwell & Chitty 1930, Dikii 1964). This assumption is true for moderate values of the parameter αR (α the wave-number and R the Reynolds number). Thus for $\alpha = 1{\cdot}0$, $\alpha R < 70$ and for $\alpha = 0{\cdot}5$, $\alpha R < 80$ with α and R defined as in §2.

It seems reasonable that there is difficulty in performing experiments on plane Couette flow. This is likely the reason why very little is known about the real behaviour of this flow. The only experiment we know that is relevant to the present problem is due to Reichardt (1956). He claims to find that Couette flow is stable for Reynolds numbers less than about 750. For larger values the flow becomes turbulent. This indicates that Couette flow is unstable for finite disturbances. It seems therefore to be an important problem to investigate the stability feature of the non-linear terms, i.e. see whether or not these are destabilizing and, most important, to examine if the effect of these terms may render the Couette flow unstable. To the authors' knowledge, the only attack on this problem is that of Kuwabara (1967). Using the same approximate equations as Meksyn & Stuart (1951) and applying a Galerkin method, he claims to find the critical Reynolds number to be about $1\cdot 9\times 10^5$. However, the validity of some of his assumptions is not discussed.

Stuart (1960) has developed a method to examine the non-linear stability behaviour of a fluid model. For periodic disturbances the amplitude $A(t)$ satisfies an equation of the Landau (1944) type,

$$\frac{d|A|^2}{dt} = k_1|A|^2 + k_2|A|^4 \tag{1.1}$$

provided that $|A|$ is sufficiently small. The terms neglected in (1.1) are of the sixth order and higher. His method has been extended by Watson (1960) and Eckhaus (1965). If the linearized solution of the problem is exponentially unstable, k_1 is positive. If in addition k_2 is negative, we notice that $|A|$ will tend towards a steady solution. For Reynolds numbers and wave-numbers sufficiently close to those on the neutral curve (if such a curve exists), k_1 is arbitrarily small and thus the solution found from (1.1) is asymptotically correct.

This method has been applied by Reynolds & Potter (1967) and Pekeris & Shkoller (1967) to investigate the non-linear stability behaviour of plane Poiseuille flow. In the first-mentioned paper the stability of a combined plane Poiseuille and Couette flow is also discussed in the cases where a neutral curve exists.

For plane Couette flow, however, the linearized solution is stable for all values of the Reynolds number and no neutral curve exists. Thus k_1 is always negative and cannot, contrary to the cases mentioned above, be made arbitrarily small for any disturbance. If k_2 is found to be negative, the considered non-linear terms also are stabilizing. On the other hand, if k_2 is positive they are destabilizing. If in this case the initial amplitude is sufficiently large, the non-linear destabilizing effect may overshadow the linear stabilizing tendency. The amplitude just large enough to render the motion unstable is obtained by finding the steady solution of (1.1). For a given wave-number this threshold amplitude depends only on the Reynolds number. This relationship constitutes a non-linear neutral curve, namely the threshold amplitude as a function of the Reynolds number for the given wave-number.

Here a serious difficulty arises. Since the steady solution so obtained is not an asymptotic solution, is it any approximation to the correct one? Or, in other words, is the steady value of $|A|$ small enough to render (1.1) a reasonable approximation to the exact equation? Important information about this question would be obtained if the next term $k_3|A|^6$ were known. The calculation of k_3 is rather involved and is not undertaken in the present paper. However, one of the typical terms contributing to k_3 is easily computed, namely the term associated with the fourth-order modification of the mean velocity profile. It is seen that if $|k_2| \gg |k_1|$, (1.1) will give $|A^2| \ll 1$. If further k_3 is of the same order of magnitude as k_2, the sixth-order term is small compared to the terms retained. This seems to be true in the present case, as far as the contribution to k_3 mentioned above is considered. This will be discussed further in § 6. It may also be a point that even though the validity of the approximations is uncertain, the threshold amplitude and the non-linear neutral curve obtained seem reasonable.

2. The solution of the linearized problem

We consider the motion of an incompressible fluid flowing between two horizontal rigid planes. The upper plane has a velocity U and the lower plane a velocity $-U$, and the distance between the planes is $2H$. Introducing U and H as the characteristic velocity and length respectively, the velocity in the undisturbed flow may be written

$$\mathbf{v}_0 = \mathbf{i}z, \tag{2.1}$$

where $\mathbf{i}$ denotes the unit vector along the horizontal x axis which is placed symmetrically between the two planes. Furthermore, let $\mathbf{v}$, ψ and ζ denote the two-dimensional perturbation velocity, the perturbation stream function and the perturbation vorticity, respectively. The equation governing the motion may then be written

$$\left(R^{-1}\nabla^2 - z\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\zeta = \mathbf{v}\cdot\nabla\zeta, \tag{2.2}$$

where

$$\zeta = \nabla^2\psi. \tag{2.3}$$

R is the Reynolds number,

$$R = UH/\nu, \tag{2.4}$$

ν the kinematic viscosity, ∇^2 the two-dimensional Laplacian, and t denotes the dimensionless time, the characteristic time being H/U. The linearized version of (2.2) reads

$$\left(R^{-1}\nabla^2 - z\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\zeta = 0. \tag{2.5}$$

The boundary conditions (no slip) lead to

$$\psi = \frac{\partial\psi}{\partial z} = 0 \quad \text{for} \quad z = \pm 1. \tag{2.6}$$

Let us assume that

$$\zeta = Af(z)\exp\{i\alpha(x-ct)\} \tag{2.7}$$

and, correspondingly,

$$\psi = AF(z)\exp\{i\alpha(x-ct)\}, \tag{2.8}$$

where A is a constant.

Equation (2.5) then takes the form

$$\left\{R^{-1}\left(\frac{d^2}{dz^2} - \alpha^2\right) - i\alpha(z-c)\right\}f(z) = 0 \tag{2.9}$$

and $F(z)$ is found from

$$\left(\frac{d^2}{dz^2} - \alpha^2\right)F(z) = f(z). \tag{2.10}$$

The boundary conditions are given by

$$F(\pm 1) = F'(\pm 1) = 0. \tag{2.11}$$

Equations (2.9) and (2.10) with the boundary conditions (2.11) constitute an eigenvalue problem with α and R given parameters and c the eigenvalue. It is readily seen from (2.9) that if $f(z)\exp\{i\alpha(x-ct)\}$ is a solution, $f(-z)\exp\{-i\alpha(x+ct)\}$ is another solution. We may therefore, instead of (2.7) and (2.8), write

$$\zeta = Af(z)\exp(\alpha c_i t)\exp\{i\alpha(x-c_r t)\} + Bf(-z)\exp(\alpha c_i t)\exp\{-i\alpha(x+c_r t)\} \quad (2.12)$$

and

$$\psi = AF(z)\exp(\alpha c_i t)\exp\{i\alpha(x-c_r t)\} + BF(-z)\exp(\alpha c_i t)\exp\{-i\alpha(x+c_r t)\}, \quad (2.13)$$

where $$c = c_r + ic_i$$

and B denotes a constant amplitude. The first part of (2.13) corresponds to a wave travelling in positive x direction and the second part to a symmetric wave travelling in the opposite direction.

Equations (2.9) and (2.10) combined may be written

$$\left\{R^{-1}\left(\frac{d^2}{dz^2}-\alpha^2\right) - i\alpha(z-c)\right\}\left(\frac{d^2}{dz^2}-\alpha^2\right)F(z) = 0. \quad (2.14)$$

The adjoint of equation (2.14) takes the form

$$\left(\frac{d^2}{dz^2}-\alpha^2\right)\left\{R^{-1}\left(\frac{d^2}{dz^2}-\alpha^2\right) - i\alpha(z-c)\right\}\tilde{F}(z) = 0, \quad (2.15)$$

where $\tilde{F}(z)$ satisfies the same boundary conditions as $F(z)$. It is easily seen that the following orthogonality relation is valid for the eigenfunctions $F_n(z)$ of (2.14) and their adjoint functions $\tilde{F}_n(z)$ (see Eckhaus 1965, p. 59)

$$\langle \tilde{F}_m(z) f_n(z)\rangle = 0 \quad (m \neq n), \quad (2.16)$$

expressing that the adjoint stream function is orthogonal to the vorticity. Correspondingly

$$\langle \tilde{F}_m(-z) f_n(-z)\rangle = 0 \quad (m \neq n). \quad (2.17)$$

Here $\langle\ \rangle$ denotes integration from $z = -1$ to $z = 1$.

If, in equation (2.9), a new variable

$$\eta = (\alpha R)^{\frac{1}{3}}(z-c) - i\kappa^2, \quad (2.18)$$

where $$\kappa = \alpha(\alpha R)^{-\frac{1}{3}},$$

is introduced, (2.9) takes the form

$$\left(\frac{d^2}{d\eta^2} - i\eta\right)f(\eta) = 0. \quad (2.19)$$

The solution of this equation may be written

$$f(\eta) = C_1 f_1(\eta) + C_2 f_2(\eta), \quad (2.20)$$

where $$f_{1,2}(\eta) = \eta^{\frac{1}{2}} H^{(1,2)}_{\frac{1}{3}}\left(\tfrac{2}{3}\eta^{\frac{3}{2}}\exp\left(-i\tfrac{1}{4}\pi\right)\right) \quad (2.21)$$

with $H^{(1)}_{\frac{1}{3}}$ and $H^{(2)}_{\frac{1}{3}}$ denoting Hankel functions of order $\frac{1}{3}$, and C_1 and C_2 are arbitrary constants. These constants are determined by the boundary conditions which also lead to the secular equation determining the eigenvalues. This secular equation has been discussed by Hopf (1914), Grohne (1954) and Riis (1962). Calculations of the eigenvalues have also been undertaken by Gallagher & Mercer (1962), applying a Galerkin method instead of making use of the eigenfunctions (2.21). The main result of these papers is that for a given value of α an infinite number of eigenvalues exist, all of them corresponding to a stable motion. In the present work we need the eigenvalues for a broader range of Reynolds number and wave-number than given in the papers referred to. We have therefore found it necessary to recalculate the eigenvalues.

The secular equation determining the eigenvalues is given in the appendix (equations (A 4)–(A 6)). In the vicinity of $\eta_{-1}\exp(-i\frac{5}{6}\pi)$, $f_2(\eta)$ is very large and rapidly oscillating. We have therefore found it necessary to integrate (A 6) by using the asymptotic expansions for the integrand. It was also found convenient to apply asymptotic expansions to integrate (A 4). The resulting equation in η_1 (and η_{-1}) was solved by a Newton–Raphson method which was found to converge very rapidly. The inaccuracy in the eigenvalues is therefore essentially due to the application of asymptotic expansions. By using a varying number of terms in the asymptotic expansion, the error in the eigenvalues is estimated to be of order 0·1 %. The result of the calculations is shown in table 1 for the first mode, i.e. the most unstable eigenfunction.

	$\alpha = 0{\cdot}5$		$\alpha = 1{\cdot}0$	
αR	c_r	$-c_i$	c_r	$-c_i$
10^3	0·5991	0·1149	0·6051	0·1196
2×10^3	0·6797	0·0899	0·6835	0·0928
3×10^3	0·7193	0·0779	0·7222	0·0802
5×10^3	0·7624	0·0652	0·7646	0·0668
10^4	0·8108	0·0513	0·8121	0·0523

TABLE 1

To tabulate the corresponding values of the vorticity and stream function we have used, for η near η_{-1}, asymptotic expansions for the Hankel functions. For other values of η we have utilized a standard routine for Bessel functions. The vorticity and stream function thus obtained are displayed in figures 1 and 2 for $\alpha = 0{\cdot}5$ and for αR equal to 10^3 and 10^4. Here the constant C_1 in (2.20) is chosen equal to unity. The adjoint stream function, tabulated and normalized in a similar way, is shown in figure 3.

It is seen from figure 1 that for high values of the Reynolds number, the vorticity is almost entirely located in the upper fluid layers, having numerical maxima in the boundary layer at the upper plane and in the layer where the velocity of propagation of the perturbation equals the basic flow. Also a concentration of vorticity is found at the lower plane.

The Reynolds stresses $\overline{uw}$ are shown in figure 4. It is seen also that $\overline{uw}$ is virtually zero except in the upper fluid layers.

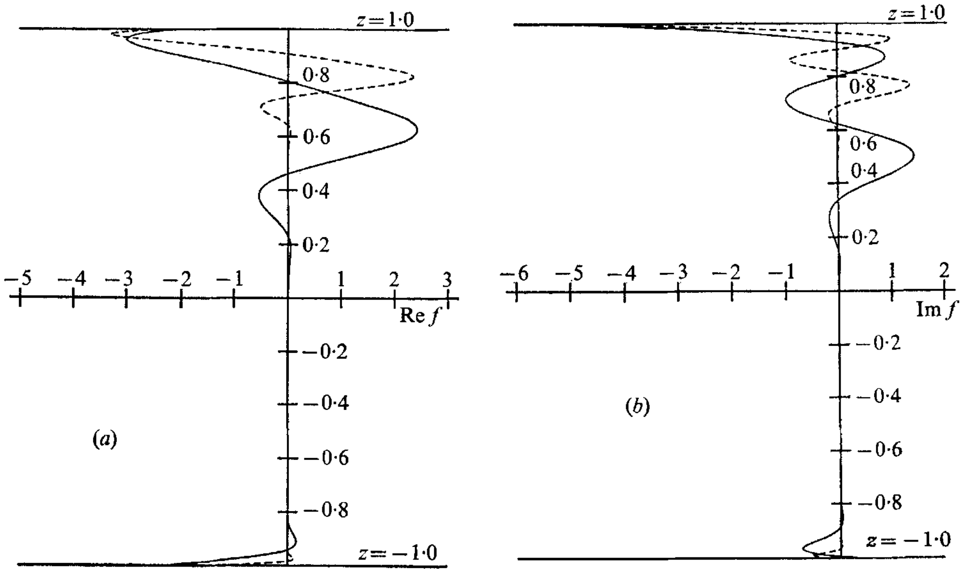

FIGURE 1. Vorticity distributions $f(z)$ for $\alpha = 0{\cdot}5$. ——, $\alpha R = 10^3$; - - - -, $\alpha R = 10^4$. (a) Real part. (b) Imaginary part.

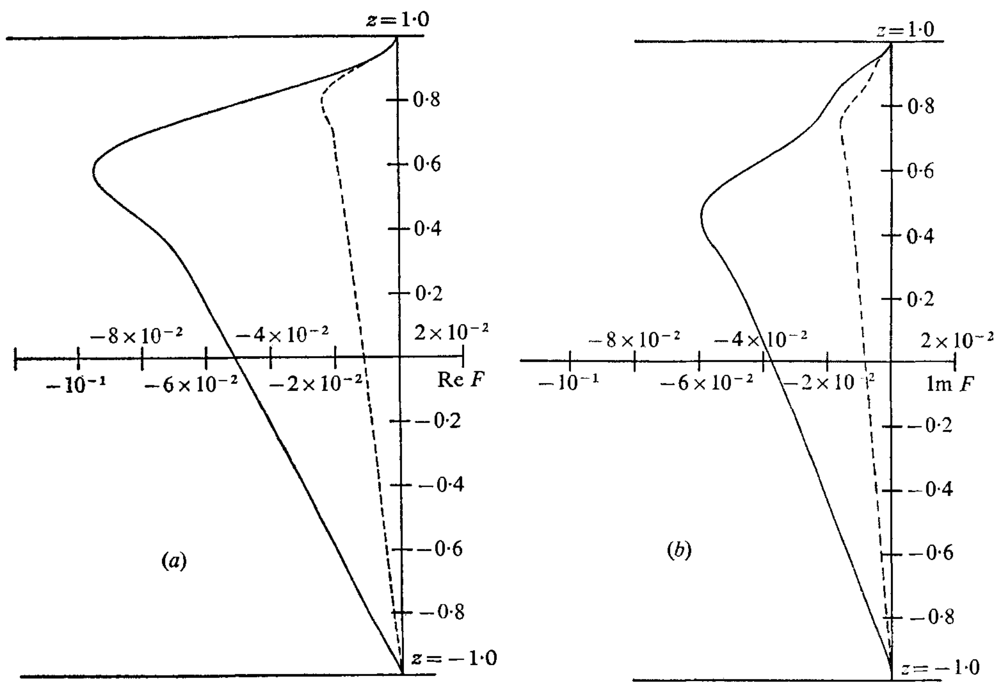

FIGURE 2. Stream function $F(z)$ for $\alpha = 0{\cdot}5$. ——, $\alpha R = 10^3$; - - - -, $\alpha R = 10^4$. (a) Real part. (b) Imaginary part.

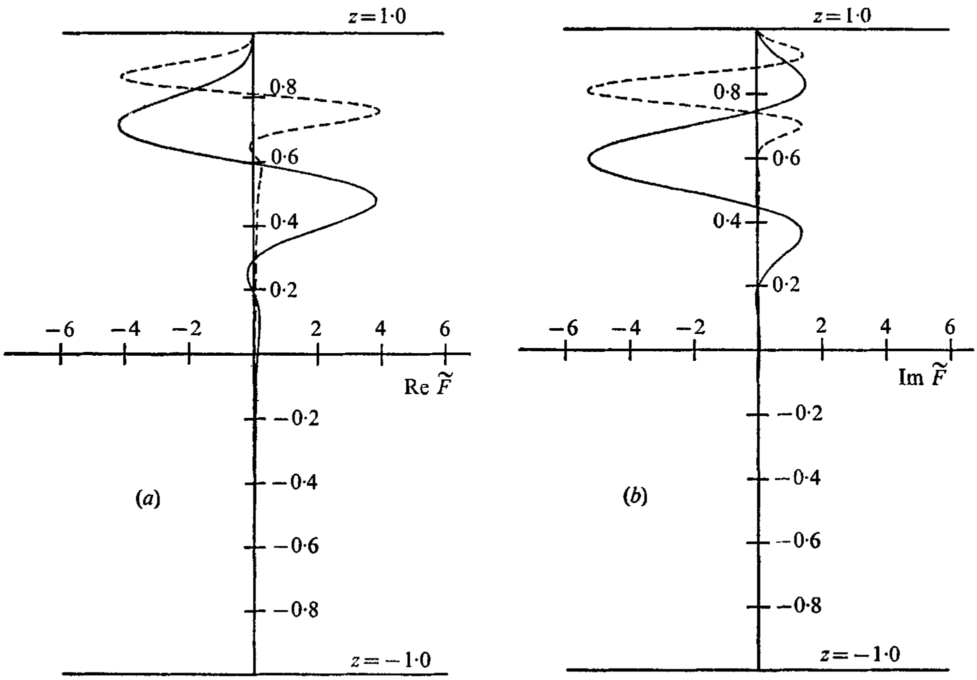

FIGURE 3. Adjoint stream function $\tilde{F}(z)$ for $\alpha = 0{\cdot}5$. ——, $\alpha R = 10^3$; - - - -, $\alpha R = 10^4$. (*a*) Real part. (*b*) Imaginary part.

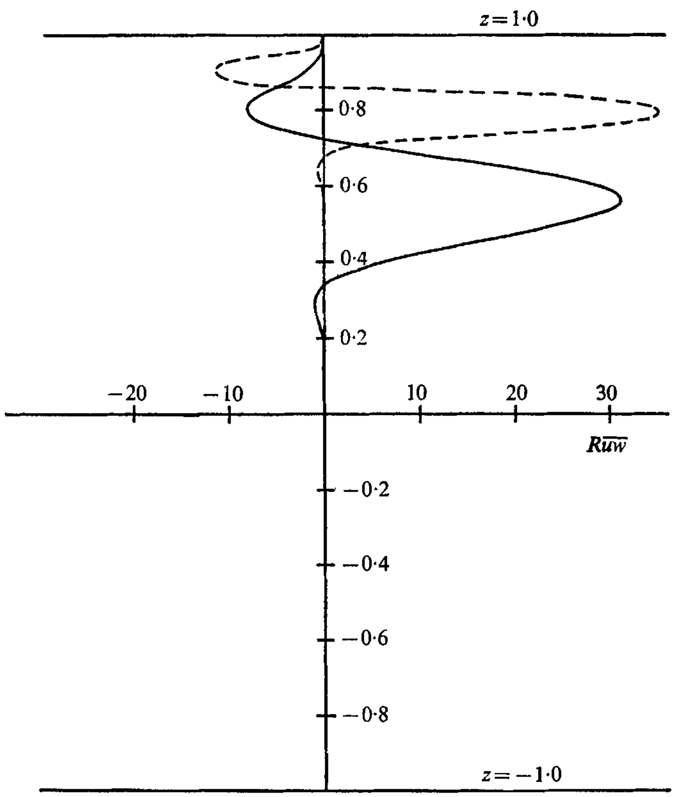

FIGURE 4. Reynolds stresses as function of z for $\alpha = 0{\cdot}5$. ——, $\alpha R = 10^3$; - - - -, $\alpha R = 10^4$.

3. The non-linear problem

It will be assumed that the perturbation is two-dimensional and for simplicity we consider only the case of a wave travelling in the positive x direction (i.e. $B = 0$). The solution of the non-linear perturbation equation will be approximated by a truncated Fourier series in the usual way. The stream function and the vorticity may be written

$$\left.\begin{aligned} \psi &= \sum_n \psi_n(z,t)\exp\{in\alpha(x-c_r t)\}, \\ \zeta &= \sum_n \zeta_n(z,t)\exp\{in\alpha(x-c_r t)\}. \end{aligned}\right\} \tag{3.1}$$

For ψ_n and ζ_n the following relations hold

$$\left.\begin{aligned} \psi_{-n} &= \psi_n^*, \\ \zeta_{-n} &= \zeta_n^*, \\ \left(\frac{\partial^2}{\partial z^2} - n^2\alpha^2\right)\psi_n &= \zeta_n, \end{aligned}\right\} \tag{3.2}$$

where an asterisk denotes the complex conjugate. By writing $\overline{u}$ for the modification of the mean flow

$$\overline{u} = \frac{\partial \psi_0}{\partial z} \tag{3.3}$$

and introducing (3.1) into (2.2), we obtain the following equations

$$\begin{aligned} \left\{R^{-1}\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right) - i\alpha(z-c_r) - \frac{\partial}{\partial t}\right\}\zeta_1 &= i\alpha\left(\zeta_1\overline{u} - \psi_1\frac{\partial^2\overline{u}}{\partial z^2}\right) \\ &\quad - i\alpha\left(\zeta_1^*\frac{\partial\psi_2}{\partial z} - \psi_1^*\frac{\partial\zeta_2}{\partial z}\right) - i2\alpha\left(\frac{\partial\zeta_1^*}{\partial z}\psi_2 - \frac{\partial\psi_1^*}{\partial z}\zeta_2\right), \end{aligned} \tag{3.4}$$

$$\left\{R^{-1}\left(\frac{\partial^2}{\partial z^2} - 4\alpha^2\right) - i2\alpha(z-c_r) - \frac{\partial}{\partial t}\right\}\zeta_2 = i\alpha\left(\zeta_1\frac{\partial\psi_1}{\partial z} - \frac{\partial\zeta_1}{\partial z}\psi_1\right), \tag{3.5}$$

$$\left(R^{-1}\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t}\right)\overline{u} = i\alpha(\zeta_1\psi_1^* - \zeta_1^*\psi_1). \tag{3.6}$$

These equations correspond to (2.5)–(2.7) in Stuart's (1960) paper, and are correct to the third order in the amplitude of ψ_1. The boundary conditions to be satisfied are

$$\overline{u} = \psi_n = \frac{\partial\psi_n}{\partial z} = 0 \quad (z = \pm 1). \tag{3.7}$$

Following Eckhaus (1965) we assume that (2.14) has a complete set of eigenfunctions $F_n(z)$ with eigenvalues c_n. ψ_1 may then be written

$$\psi_1(z,t) = \sum_{n=1}^{\infty} A_n(t)F_n(z). \tag{3.8}$$

Let c_1 denote the eigenvalue for the least stable mode (i.e. minimum αc_i) for the given values of α and R. We then assume that ψ_1 to the first approximation is given by

$$\psi_1(z,t) = A(t)F(z), \tag{3.9}$$

where for simplicity we have neglected subscripts by writing $A(t)F(z)$ for $A_1(t)F_1(z)$. The assumption involved in (3.9) is generally made in related problems, and corresponds to an intuitive reasonable choice of the initial conditions.

When (3.8) is introduced into (3.4)–(3.6) the approximation (3.9) can be used on the right-hand sides of these equations. Further the orthogonality relation (2.16) is applied on (3.4) and the following equation is then obtained

$$(\dot{A}(t)-\alpha c_i A(t))\langle f(z)\tilde{F}(z)\rangle = -\langle M(z,t)\tilde{F}(z)\rangle, \tag{3.10}$$

where

$$M(z,t) = i\alpha A\left(f\overline{u}-F\frac{\partial^2\overline{u}}{\partial z^2}\right)-i\alpha A^*\left(f^*\frac{\partial\psi_2}{\partial z}-F^*\frac{\partial\zeta_2}{\partial z}\right)-i2\alpha A^*(f'^*\psi_2-F'^*\zeta_2). \tag{3.11}$$

$\overline{u}$, ζ_2 and ψ_2 are now to be determined from

$$\left(R^{-1}\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial t}\right)\overline{u} = i\alpha AA^*(fF^*-f^*F), \tag{3.12}$$

$$\left\{R^{-1}\left(\frac{\partial^2}{\partial z^2}-4\alpha^2\right)-i2\alpha(z-c_r)-\frac{\partial}{\partial t}\right\}\zeta_2 = i\alpha A^2(fF'-f'F), \tag{3.13}$$

$$\left(\frac{\partial^2}{\partial z^2}-4\alpha^2\right)\psi_2 = \zeta_2. \tag{3.14}$$

Here $f(z)$ and $\tilde{F}(z)$ are the vorticity and adjoint eigenfunction, corresponding to $F(z)$.

The solutions of the equations (3.10)–(3.14) are very difficult to obtain in the general case. However, if $\dot{A}/A$ is approximately constant, one solution of (3.12)–(3.14) can easily be found by separation of the variables. This is the case if (i) A is so small that $\dot{A}/A$ is approximately equal to αc_i, or if (ii) AA^* is nearly independent of time. In both cases we can write

$$\left.\begin{aligned}\overline{u} &= i\alpha AA^*u(z),\\ \zeta_2 &= i\alpha A^2 g(z),\\ \psi_2 &= i\alpha A^2 G(z)\end{aligned}\right\} \tag{3.15}$$

and obtain an amplitude equation of the Landau type

$$\dot{A}-\alpha c_i A+\alpha^2 pA^2A^* = 0, \tag{3.16}$$

where $p = p_r+ip_i$ is defined by

$$p\langle f\tilde{F}\rangle = \langle\tilde{F}(Fu''-fu-F^*g'-2F'^*g+f^*G'+2f'^*G)\rangle. \tag{3.17}$$

By combination of (3.16) and its complex conjugate the equation

$$\frac{d}{dt}|A|^2 = k_1|A|^2+k_2|A|^4 \tag{3.18}$$

is obtained. The coefficients are given by

$$\left.\begin{aligned}k_1 &= 2\alpha c_i,\\ k_2 &= -2\alpha^2 p_r.\end{aligned}\right\} \tag{3.19}$$

4. The amplitude equation for small amplitudes

If the amplitude is sufficiently small, the truncated Fourier series will be a good approximation. In this case we also have $\dot{A}/A$ approximately equal to αc_i and from (3.11)–(3.14) we therefore get

$$\left(R^{-1}\frac{d^2}{dz^2} - 2\alpha c_i\right) u = fF^* - f^* F, \tag{4.1}$$

$$\left\{R^{-1}\left(\frac{d^2}{dz^2} - 4\alpha^2\right) - i2\alpha(z - c_r) - 2\alpha c_i\right\} g = fF' - f'F, \tag{4.2}$$

$$\left(\frac{d^2}{dz^2} - 4\alpha^2\right) G = g. \tag{4.3}$$

If in these equations αc_i is put equal to zero, the equations (4.3) and (4.4) in Stuart's (1960) paper are obtained. Stuart cancels the terms containing αc_i since in his paper they are small of the fourth order. Here, however, αc_i cannot be chosen arbitrary small, and is therefore retained. It may, true enough, be argued that for large values of R (which we are concerned with) αc_i tends towards zero as $(\alpha R)^{-\frac{1}{3}}$. However, also the term $R^{-1}d^2/dz^2$ becomes small of the order $(\alpha R)^{-\frac{1}{3}}$ for increasing R.

Solving (4.1)–(4.3) with the appropriate boundary conditions we obtain the second-order approximation. It is seen that the solution of (4.2) is composed of Bessel functions of order $\frac{1}{3}$, for details, see appendix B. When these solutions are introduced in (3.17), p and thereby k_2 in (3.18) is found by an integration. It is found that the sign of k_2 changes quite rapidly when α and R vary. As an example we may choose $\alpha = 0{\cdot}5$. For $\alpha R = 10^3$ and 3×10^3 we find $k_2 = 8{\cdot}01$ and $7{\cdot}13$, respectively (destabilizing). For $\alpha R = 2 \times 10^3$ and 5×10^3 we find $k_2 = -21{\cdot}19$ and $-1{\cdot}14$ respectively (stabilizing).

It must here, however, be pointed out that $\bar{u}$ is initially not zero. This means that we are really examining the stability properties of a velocity profile which is not strictly a Couette profile. The results seem therefore only to indicate that for some velocity profiles close to the Couette profile the second-order terms act stabilizing whereas for other profiles they act destabilizing. It may here be mentioned that the solution of (4.1) becomes infinite when $\alpha Rc_i = -\frac{1}{8}n^2\pi^2$. This means that in these cases $\bar{u}$ is initially infinite. Therefore, in order that the solution shall be of any physical meaning, c_i must not have a value close to one of these critical values.

It is of much more interest to consider the case that $\bar{u}$ is initially zero. This does not, however, lead to any amplitude equation of the Landau type since $\bar{u}$ is not proportional to AA^*. If, however, ζ_2 (and ψ_2) are given initial values in agreement with (3.15) and only small values of time are considered, an amplitude equation of the former type will occur. Under these conditions $\bar{u}$ may be omitted, and the new value of k_2 is found from that above by neglecting the contributions from $\bar{u}$ in (3.17).

It is now found that for all values of α and R considered (α equal to $0{\cdot}5$ and $1{\cdot}0$ and αR in the range 10^3–10^4) k_2 is positive. This shows that at least for small values of time the non-linear terms act destabilizing.

5. The equation for the threshold amplitude

The results of the previous section indicate that the non-linear terms may decrease the stability of the Couette flow. The important problem is, however, whether for sufficiently large disturbances, the non-linear destabilizing tendency may overcome the linear stabilizing effect such that the motion becomes unstable. If this is true, a threshold amplitude must exist, i.e. an amplitude dependent of α and R for which the disturbance is neutral. For such a disturbance $A(t)$ is a harmonic function of time, $A(t) \propto \exp(-i\alpha\Delta c_r t)$ where Δc_r is the change in the wave velocity due to the non-linear terms.

Assuming that it suffices to take into consideration terms to the third order only, the calculations will be very similar to those given in §4. With the time dependence assumed and by neglecting Δc_r, (3.12) and (3.13) lead to

$$R^{-1}\frac{d^2u}{dz^2} = fF^* - f^*F, \tag{5.1}$$

$$\left\{R^{-1}\left(\frac{d^2}{dz^2} - 4\alpha^2\right) - i2\alpha(z - c_r)\right\}g = fF' - f'F, \tag{5.2}$$

instead of (4.1) and (4.2).

At this stage Δc_r is an unknown quantity and we have found no better approximation than putting Δc_r equal to zero. An estimate of Δc_r may be found from (3.16) which gives

$$\Delta c_r = c_i p_i/p_r. \tag{5.3}$$

If now p_i and p_r are computed from the equations above (with $\Delta c_r = 0$) it is found that p_i/p_r is very close to unity for the considered values of α and R. From table 1 it is then seen that for large values of αR, Δc_r is small compared to c_r. The main effect of taking into account Δc_r in (5.2) would be to displace the critical layer for ζ_2 compared to that of ζ_1. Since ζ_1 and ζ_2 are moving with the same velocity of propagation, their critical layers should be located at the same level. This can be obtained in two ways, either by taking into account Δc_r in the definition of ζ_1 (obviously being the most correct way but leading to considerable difficulties), or putting $\Delta c_r = 0$ in ζ_2, as we have done here. It seems reasonable that these two ways of proceeding will lead to approximately the same values for p.

The equations (5.1) and (5.2) are similar to those applied by Stuart (1960). The reasoning for deriving the equations are, however, somewhat different, since Stuart cancels $\partial/\partial t$ in the equations corresponding to (3.12) and (3.13) by arguing that it leads to higher order terms.

The coefficient k_2 in (3.18) may now be found by applying the solutions of (5.1) and (5.2) in (3.17). It is found that k_2 is negative for the values of α and R considered, such that an threshold amplitude can be found. The neutral curve is shown in figure 5 for $\alpha = 0{\cdot}5$ and $\alpha = 1{\cdot}0$ and for αR in the range 10^3–10^4.

6. Discussion and conclusion

The main conclusion to be drawn from §4 is that only in special cases does an amplitude equation of the Landau type exist. It is found that if $\bar{u}$ is initially zero and ψ_2 is initially given by (3.15), an amplitude equation exists for small values of time t. The non-linear terms are found to act destabilizing. If, however, $\bar{u}$ is initially different from zero, the non-linear term will act stabilizing or destabilizing, depending on the initial velocity profile.

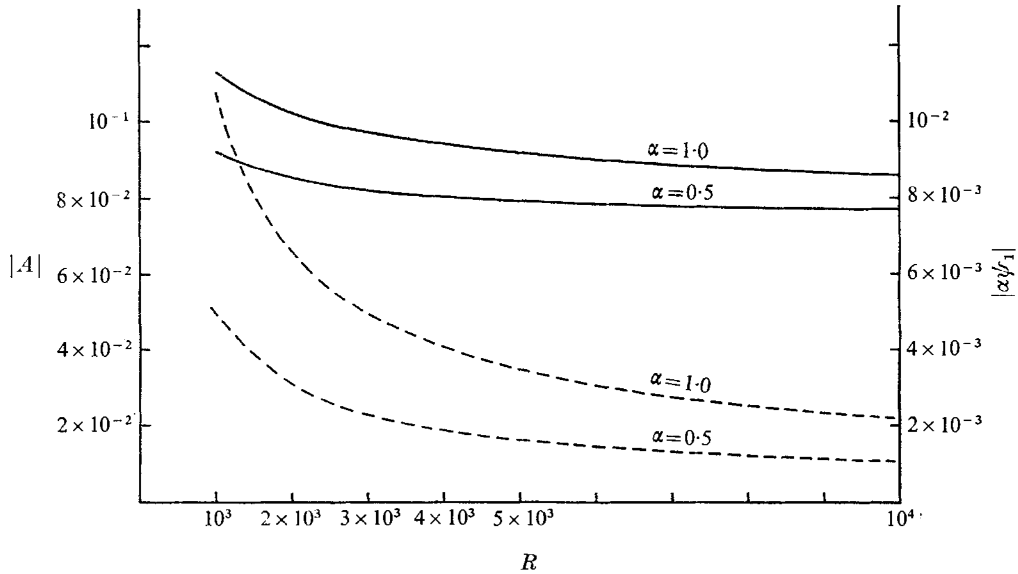

FIGURE 5. Amplitudes of the neutral solution as function of R. ——, the vorticity amplitude $|A|$; – – – –, the maximum vertical velocity $|\alpha\psi_1|$.

The principal result from §5 is the existence of a threshold amplitude. It should be noted that since the vorticity distribution $f(z)$ is normalized to be of order unity for all values of α and R, the amplitude $|A|$ displayed in figure 5 is the threshold amplitude for the vorticity. It is seen that $|A|$ decreases very slowly as αR increases, and the curve suggests a constant value of the amplitude for large αR. This result is also suggested from a closer inspection of the vorticity equation. By introducing η, defined by (2.18), the vertical co-ordinate is stretched such that for large values of αR, $\partial/\partial\eta$ is of order unity. Therefore, the horizontal and vertical velocities will be of order $(\alpha R)^{-\frac{1}{3}}$ and $(\alpha R)^{-\frac{2}{3}}$ times that of the vorticity, respectively. Introducing this in (2.2), we end up with an asymptotic equation for ζ which is independent of αR.

In figure 5 is also shown the maximum value (occurring near the critical layer) of the vertical velocity $i\alpha\psi_1$, which accordingly decays as $(\alpha R)^{-\frac{2}{3}}$. This velocity is small compared to the velocity of the planes which is unity. It is found that for $\alpha R = 10^3$ and 10^4, the modulus of $i\alpha\psi_1$ for $\alpha = 0\cdot5$ is 5×10^{-3} and 10^{-3} respectively.

If the initial perturbation is concentrated in a layer about $z = z_0$, the initial stream function may be approximated by a δ function, $\psi_1(z,0) = a\delta(z-z_0)$. Applying an eigenfunction expansion and assuming that the higher order eigenfunctions do not influence the amplitude of the first eigenfunction essentially (since they are more stable), we obtain from the orthogonality relations (2.16)

$$a\{\tilde{F}''(z_0) - \alpha^2 \tilde{F}(z_0)\} = A(0)\langle \tilde{F}(z) f(z)\rangle. \tag{6.1}$$

If z_0 is chosen as the position of the critical layer, it is found that the amplitude $|a|$ is $2{\cdot}4 \times 10^{-3}|A(0)|$ for $\alpha R = 10^3$. It is noted that this amplitude is much smaller than that of the maximum vertical velocity discussed above. If z_0 is placed outside the critical layer, the amplitude a is found to increase radically.

It may be worth while to mention that the Meksyn–Stuart approximation (Meksyn & Stuart 1951) would have given the correct sign of the coefficient k_2 in (3.18). This is in accordance with the result of Reynolds & Potter (1967) for plane Poiseuille flow. In the present problem, however, it turns out that the contributions from the terms proportional to $\exp(2i\alpha x)$ are just as important as the contributions from $\overline{u}$ and $\partial^2\overline{u}/\partial z^2$. For example for $\alpha = 0{\cdot}5$ and $\alpha R = 10^3$ the contribution to k_2 from the terms of the latter type is $4{\cdot}65$ whereas that from terms proportional to $\exp(2i\alpha x)$ is $8{\cdot}91$. Correspondingly, for $\alpha R = 10^4$ we find the values $2{\cdot}17$ and $6{\cdot}71$, respectively.

An estimate of the magnitude of the terms neglected in the derivation above is very difficult to obtain. It is here pertinent to refer to a suggestion of Stuart (1960) that to secure the validity of the expansion, $c_i \ll (\alpha R)^{-\frac{1}{3}}$ must hold. Stuart's suggestion is based on the argument that Δc_r must be much smaller than the width of the critical layer, in order that the critical layers of the first approximation and the finite solution shall nearly coincide. By using a line of argument as in §5, it seems to us that this requirement is too severe, at least when only a first approximation to the amplitude equation is wanted.

The result found above that the threshold amplitude is asymptotically independent of αR, is reflected in the coefficients $k_2, k_3, \ldots$, in the amplitude equation, in fact all these constants will be asymptotically proportional to $(\alpha R)^{-\frac{1}{3}}$. The convergence of the series is therefore independent of the Reynolds number when $\alpha R \to \infty$. A test of the convergence may be obtained by calculating the term $k_3|A|^6$. The ratio k_2/k_1 is found to be large, rendering $|A|$ small in the first approximation. If $|k_3| \ll k_2^2/|k_1|$, $k_3|A|^6$ obviously may be cancelled. To determine k_3 it is necessary to know the third-order terms.

These terms are not computed here; there is, however, one part of k_3 which can be readily found. This contribution, k_3', is due to the change of the mean flow caused by waves of wave-number 2α, and is believed to be an important one. It is found that for $\alpha = 0{\cdot}5$ and $\alpha R = 10^3$, $k_3' = -118{\cdot}7$, while for $\alpha R = 10^4$, $k_3' = -130{\cdot}9$. The ratio $|k_3'|\,|A|^6/|k_2|\,|A|^4$ is thus about $7{\cdot}4 \times 10^{-2}$ and $8{\cdot}5 \times 10^{-2}$ for $\alpha R = 10^3$ and 10^4, respectively, which indicates a reasonable good convergence of the series.

On the other hand, the energy equation indicates a rather poor convergence. In the neutral case the work of the Reynolds stresses must be negative to compensate the dissipation. In our first approximation (the linear solution), however, this work is found to be positive and in fact nearly equal to the dissipation, indicating that the Couette flow exhibits relatively strong (linear) stability. The work of the Reynolds stresses due to higher order terms must therefore be at least twice that of the linear terms in order that a balance in the energy equation should be obtained. We therefore believe that our amplitude equation gives only a relatively rough approximation to the threshold amplitude.

The numerical calculation was carried out on the CDC 3300 computer of the University of Oslo. In §2 is described the method used in tabulating the eigenvalues and the first-order solutions. The same method is also used for the second-order solutions, quoted in appendix B. The integrals in (3.17) are then obtained by a numerical integration. The second-order solutions show many of the same features as the first-order solutions, and it is found that the lower half of the field gives no noticeable contribution to the integrals. The length of the subintervals was chosen equal to 5×10^{-3} and the accuracy was checked in some few cases by doubling the number of subintervals. It turns out that the integrands are very sensitive to inaccuracies in the eigenvalues. The error in k_2 is however estimated to be less than 5 %.

The authors wish to thank their colleague Dr E. Riis for many valuable discussions.

Appendix A

The first-order vorticity $f(\eta)$ is given by (2.20). Since the stream function may be written

$$F = \alpha^{-1}(\alpha R)^{-\frac{1}{3}}\int_{\eta_{-1}}^{\eta} f(u)\sinh\kappa(\eta-u)\,du, \tag{A 1}$$

the secular equation takes the form

$$\int_{\eta_{-1}}^{\eta_1} f_1(u)\exp(\kappa u)\,du\int_{\eta_{-1}}^{\eta_1} f_2(u)\exp(-\kappa u)\,du \\ -\int_{\eta_{-1}}^{\eta_1} f_1(u)\exp(-\kappa u)\,du\int_{\eta_{-1}}^{\eta_1} f_2(u)\exp(\kappa u)\,du = 0. \tag{A 2}$$

κ and η are defined in §2, and $\eta_{\pm 1} = \eta(z = \pm 1)$ are complex numbers, located in the first and second quadrants, respectively.

Following Riis (1962), we transform the integrals in the following way. First we write

$$\int_{\eta_{-1}}^{\eta_1}(\ )\,du = \int_{\eta_{-1}}^{0}(\ )\,du + \int_{0}^{\eta_1}(\ )\,du.$$

By means of the relations

$$\left.\begin{aligned} f_1(u) &= -f_2(u\exp(-i\tfrac{2}{3}\pi)),\\ f_2(u) &= \exp(i\tfrac{1}{3}\pi)f_2(u\exp(-i\tfrac{2}{3}\pi))\\ &\quad -\exp(-i\tfrac{1}{3}\pi)f_1(u\exp(-i\tfrac{2}{3}\pi)),\end{aligned}\right\} \tag{A 3}$$

it is found that, in all the integrals, the path of integration will be located in the region $-\frac{1}{6}\pi \leqslant \arg u \leqslant \frac{1}{2}\pi$. For large values of αR, $f_{1,2}(\eta_{-1}\exp(-i\frac{2}{3}\pi))$ will be large compared with $f_1(\eta_1)$. Utilizing this, (A 2) may with a good approximation be written

$$\exp(i\tfrac{1}{3}\pi)\int_{\infty\exp(i\frac{1}{2}\pi)}^{\eta_1} f_1(u)\{\exp(\kappa u)-\mu\exp(-\kappa u)\}\,du+\int_{\infty\exp(-i\frac{1}{6}\pi)}^{\eta_1} f_2(u) \times\{\exp(\kappa u)-\mu\exp(-\kappa u)\}\,du+C(\kappa)-\mu C(-\kappa)=0, \quad (\mathrm{A}\,4)$$

where (see Riis 1962, appendix A)

$$C(\kappa)=\exp(i\tfrac{1}{3}\pi)\int_0^{\infty\exp(i\frac{1}{2}\pi)} f_1(u)\{\exp(\kappa u\exp(i\tfrac{2}{3}\pi)) +\exp(\kappa u)+\exp(\kappa u\exp(-i\tfrac{2}{3}\pi))\}\,du=2\cdot 3^{\frac{1}{2}}\exp\{(i/12)(5\pi-4\kappa^3)\} \quad (\mathrm{A}\,5)$$

and μ is a constant given by

$$\int_{\infty\exp(-i\frac{1}{6}\pi)}^{\eta_{-1}\exp(-i\frac{2}{3}\pi)} f_2(u)\exp(\kappa u\exp(i\tfrac{2}{3}\pi))\,du =\mu\int_{\infty\exp(-i\frac{1}{6}\pi)}^{\eta_{-1}\exp(-i\frac{2}{3}\pi)} f_2(u)\exp(-\kappa u\exp(i\tfrac{2}{3}\pi))\,du. \quad (\mathrm{A}\,6)$$

Appendix B

By introducing $\omega^2=-2\alpha Rc_i$, the solution of (4.1) takes the form

$$u(z)=\frac{R}{\omega\sin 2\omega}\left\{\sin 2\omega\int_{-1}^{z}(fF^*-f^*F)\sin\omega(z-u)\,du -\sin\omega(z+1)\int_{-1}^{+1}(fF^*-f^*F)\sin\omega(1-u)\,du\right\}. \quad (\mathrm{B}\,1)$$

The solution of (5.1) is then obtained from (B 1) as the limit $\omega\to 0$. For the stream function G we have

$$G(z)=(2\alpha)^{-1}\int_{-1}^{z} g(u)\sinh 2\alpha(z-u)\,du. \quad (\mathrm{B}\,2)$$

Let $R_1=\alpha^{-1}(\alpha R)^{\frac{1}{3}}(fF'-f'F)$ be considered as a function of η (by means of (2.18)). The solution of (4.2) is then

$$g(z)=\Delta^{-1}\left\{D_1\phi_1(\eta)+D_2\phi_2(\eta) +\phi_1(\eta)\int_{\eta_1}^{\eta}\phi_2(u)\,R_1(u)\,du-\phi_2(\eta)\int_{\eta_{-1}}^{\eta}\phi_1(u)\,R_1(u)\,du\right\}. \quad (\mathrm{B}\,3)$$

$\phi_1(\eta)$ and $\phi_2(\eta)$ are fundamental solutions of

$$\left\{\frac{d^2}{d\eta^2}-2i(\eta-i\kappa^2)\right\}\phi(\eta)=0$$

and Δ is the Wronskian

$$\Delta=\phi_1'\phi_2-\phi_1\phi_2'=2^{\frac{1}{3}}(6i/\pi).$$

A convenient choice of ϕ_1 and ϕ_2 is

$$\left.\begin{aligned}\phi_1(\eta) &= f_1(2^{\frac{1}{3}}(\eta - i\kappa^2)) + \exp(-i\tfrac{1}{3}\pi) f_2(2^{\frac{1}{3}}(\eta - i\kappa^2)),\\ \phi_2(\eta) &= f_2(2^{\frac{1}{3}}(\eta - i\kappa^2)).\end{aligned}\right\} \tag{B 4}$$

The solution of (5.2) has the same form (B 3). In fact, the only modification needed is to replace κ^2 in (B 4) by $\kappa^2 - c_i(\alpha R)^{\frac{1}{3}}$.

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